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Note

At least three minimal quasi-kernels

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ABSTRACT

If D is a digraph, then $K \subseteq V(D)$ is a quasi-kernel of D if K is independent and for each $y \in V(D) - K$ there is $x \in K$ such that the directed distance from y to x is less than three. Note that any independent superset of a quasi-kernel is a quasi-kernel. Jacob and Meyniel have given a sufficient condition for a digraph to have at least three quasi-kernels, however these quasi-kernels need not be minimal. We give sufficient conditions for a digraph to have at least three minimal quasi-kernels.

Notation. For a digraph D, $V(D)$ and $A(D)$ denote its vertex set and arc set, respectively. If $U \subseteq V(D)$, then $D[U]$ denotes the subdigraph of D induced by U and $\mathrm{in}(U)$ and $\mathrm{out}(U)$ denote, respectively, the in- and out- neighborhoods of U. If $U = \{x\}$, these latter sets may be written as $\mathrm{in}(x)$ and $\mathrm{out}(x)$.

Definition 1. A quasi-kernel K of a digraph D is a subset of $V(D)$ satisfying two properties:

(1) the quasi-absorbing property: $\forall y \in V(D) - K$, $\exists x \in K$ such that the directed distance in D from y to x is either one or two;
(2) independence: there is no arc in $A(D)$ between vertices of K.

If K is a quasi-kernel for digraph D, $x \in K$ and $y \in V(D) - K$ and the directed distance from y to x in D is either one or two, then we will say that x quasi-absorbs y.

By [2], every digraph has at least one quasi-kernel. Clearly, any independent superset of a quasi-kernel is a quasi-kernel. We have chosen to focus on minimal quasi-kernels (i.e. quasi-kernels that do not contain other quasi-kernels). In [4], Heard and Huang show that a semicomplete multipartite digraph with no kernel has at least three *disjoint* quasi-kernels and in [5], Jacob and Meyniel prove that every digraph without a kernel possesses at least three distinct quasi-kernels, but in their proof the quasi-kernels need not be minimal. For example, the digraph in Fig. 1 has no kernel and four quasi-kernels, but only one of them is minimal.

In Theorem 1 we prove that the exclusion of sinks, 2-cycles and 4-cycles from a digraph is sufficient to ensure that it has at least three minimal quasi-kernels. In [3], Theorem 2.2, Gutin et al. give necessary and sufficient conditions of a digraph without sinks to have exactly two quasi-kernels. The implication is that a digraph without sinks which fails their conditions has at least three quasi-kernels. However, again, not all of these quasi-kernels must be minimal. For example, if one adds $5 \to 1$ to the 4-cycle $1 \to 2 \to 3 \to 4 \to 1$, one obtains a digraph failing the conditions for Theorem 2.2. of [3] and having exactly three quasi-kernels, but only two minimal ones. Theorem 2 and the subsequent comment extend further the set of digraphs guaranteed to have at least three minimal quasi-kernels.

Lemma 1. *If s and t are vertices of a digraph D such that each has an out-neighbor outside $\{s, t\}$, then D has a quasi-kernel containing neither s nor t.*

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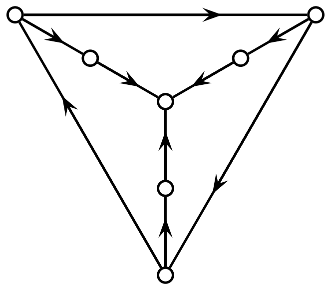

Fig. 1. No 2-cycle and no 4-cycle and exactly one minimal quasi-kernel.

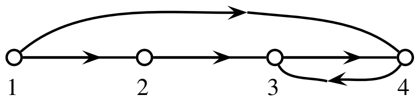

Fig. 2. No sink and no 4-cycle and exactly two minimal quasi-kernels.

Proof. Let $z_1 \in \text{out}(s) - \{t\}$ and $z_2 \in \text{out}(t) - \{s\}$. Suppose first that D has an arc $z_2 \to z_1$ and let $R = V(D) - \{z_1\} - \text{in}(z_1) - \text{in}(z_2)$. If $R = \emptyset$ then $\{z_1\}$ is the desired quasi-kernel, so suppose that R is non-empty and let $\tilde{K}$ be a quasi-kernel of $D[R]$. If $z_1 \notin \text{in}(\tilde{K})$, then $\tilde{K} \cup \{z_1\}$ is the desired quasi-kernel. Otherwise either $z_2 \in \text{in}(\tilde{K})$, in which case $\tilde{K}$ is the desired quasi-kernel, or $z_2 \notin \text{in}(\tilde{K})$, in which case $\tilde{K} \cup \{z_2\}$ is the quasi-kernel we seek.

Thus it suffices to assume that $B = \{z_1, z_2\}$ is independent. Let $R = V(D) - B - \text{in}(B)$. If $R = \emptyset$, then B is the desired quasi-kernel, so assume $R \neq \emptyset$, let $\tilde{K}$ be a quasi-kernel of $D[R]$, and set $K = \tilde{K} \cup (B - \text{in}(\tilde{K}))$. K is independent by construction (and the assumption that B is). Note moreover that

$$V(D) - K = (R - \tilde{K}) \cup \text{in}(\tilde{K}) \cup \text{in}(B).$$

Each vertex of $(R - \tilde{K}) \cup \text{in}(\tilde{K})$ is quasi-absorbed by a vertex of $\tilde{K}$ and each vertex of in(B) either beats a vertex of $B - \text{in}(\tilde{K})$ or beats a vertex of $B \cap \text{in}(\tilde{K})$ (and so is quasi-absorbed by a vertex of $\tilde{K}$). It follows that K is a quasi-kernel of D and clearly s and t lie outside K. $\square$

Comment. *Note that the lemma does not assume that s and t are distinct.*

Theorem 1. *If D is a digraph with no sinks, no 2-cycles and no 4-cycles, then D has at least three minimal quasi-kernels.*

Proof. By Theorem 2 from [1] (restated for the converse digraph), D has at least two minimal quasi-kernels, say K_1 and K_2. The general approach of the proof is to demonstrate the existence of a quasi-kernel which fails to contain at least one element of K_1 and at least one element of K_2. Such a quasi-kernel contains a minimal quasi-kernel which is necessarily distinct from K_1 and K_2.

If $u \in K_1 \cap K_2$, then the lemma provides a quasi-kernel not containing u. So suppose that $K_1 \cap K_2 = \emptyset$. If there are vertices $x \in K_1$ and $y \in K_2$ which are not adjacent, then the lemma assures the existence of a quasi-kernel containing neither x nor y and again we are done. Thus, for the rest of the proof we will assume, by the Lemma, that

$$\begin{aligned}&\text{each pair } \{s, t\} \text{ of vertices from distinct quasi-kernels are adjacent}\\ &\quad \text{and, if } s \to t, \text{ then } s \text{ has no other successor.}\end{aligned} \tag{$\star$}$$

Suppose that $|K_1| \geq 2$ and $|K_2| \geq 2$, that $x_1, x_2 \in K_1$ and $y_1, y_2 \in K_2$ and, for concreteness, that $x_1 \to y_1$. By successive use of ($\star$) it follows that $y_2 \to x_1$, that $x_2 \to y_2$, and that $y_1 \to x_2$. This contradicts our assumption that D has no 4-cycle.

Assume next that $K_1 = \{x_1, \ldots, x_n\}$ with $n \geq 2$ and $K_2 = \{y\}$.

(1) Suppose that y beats a vertex of K_1, say x_1. By ($\star$), y is beaten by $x_2, \ldots, x_n$. y must quasi-absorb x_1 (but is not beaten by it) so there exists a vertex w such that $x_1 \to w \to y$. No vertex of in(y) is beaten by w (otherwise D has a 4-cycle) and w is not beaten by $x_2, \ldots, x_n$ (by $\star$, their only successor is y). Thus $Q = \{w, x_2, \ldots, x_n\}$ is independent. Let $R = V(D) - Q - \text{in}(Q) - \text{in}(\text{in}(Q))$. If R is empty, then Q is a quasi-kernel of D containing neither x_1 nor y, so suppose $R \neq \emptyset$. Clearly, no $z \in R$ is quasi-absorbed by any vertex of Q and, in particular, such a z is not an element of in(x_1), since then it would be quasi-absorbed by w. It follows, since K_1 satisfies the quasi-absorbing property, that $R \subseteq \text{in}(\text{in}(x_1))$. Since D is assumed to have no 4-cycle, no vertex of R is beaten by w and since y is the only out-neighbor of $x_2, \ldots, x_n$, it follows that nothing in R is beaten by anything in Q. Thus, if $\tilde{K}$ is a quasi-kernel of $D[R]$, then $Q \cup \tilde{K}$ is a quasi-kernel of D containing neither x_1 nor y.

(2) Suppose that y beats no vertex of K_1. By ($\star$), y is beaten by every vertex in K_1 and, on the other hand, some vertex, say x_1 must quasi-absorb y, so there exists $w \in V(D) - K_1 - K_2$ such that $y \to w \to x_1$. This time let $R = V(D) - \{w\} - \text{in}(w) - \text{in}(\text{in}(w))$. If $R = \emptyset$, then $\{w\}$ is a quasi-kernel of D distinct from K_1 and K_2, so assume $R \neq \emptyset$. Clearly $R \subseteq \text{in}(\text{in}(y)) - \{w\} = \tilde{R}$. No vertex of $\tilde{R}$ is beaten by w (since D has no 4-cycle) and w is not beaten by any vertex of R. Thus, if $\tilde{K}$ is a quasi-kernel of $D[R]$, then $\tilde{K} \cup \{w\}$ is a quasi-kernel of D which contains neither x_1 nor y.

Finally, assume that $K_1 = \{x\}$ and $K_2 = \{y\}$. By ($\star$) we may assume that $x \to y$, but x must quasi-absorb y, so there exists w such that $y \to w \to x$. Set $R = V(D) - \{w\} - \text{in}(w) - \text{in}(\text{in}(w))$. If $R = \emptyset$, then $\{w\}$ is the desired quasi-kernel, so assume $R \neq \emptyset$ and let $\tilde{K}$ be a quasi-kernel of $D[R]$. Since $y \in \text{in}(w)$, it follows that

$$V(D) - (\{y\} \cup \text{in}(y)) \supseteq V(D) - (\text{in}(w) \cup \text{in}(\text{in}(w))) \supseteq R.$$

Thus, $R \subseteq \text{in}(\text{in}(y)) - \{w, x, y\}$. Since D has no 4-cycle, no vertex of R is beaten by w. By definition of R, w is beaten by no vertex of R, so $\tilde{K} \cup \{w\}$ is independent and thus is the quasi-kernel required to complete the proof. □

We note that no one of the conditions of Theorem 1 can be eliminated without losing the theorem, as can be seen from the following three examples.

(1) A 4-cycle has no sink and no 2-cycle and only two (minimal) quasi-kernels.
(2) The digraph in Fig. 1 has no 2-cycle and no 4-cycle and exactly one minimal quasi-kernel.
(3) The digraph in Fig. 2 has no sink and no 4-cycle and exactly two minimal quasi-kernels (viz. {3} and {4}).

The closing theorem summarizes our findings.

Theorem 2. *Let S be the set of sinks in a digraph D.*

(1) *D has a unique quasi-kernel if and only if $V(D) = S \cup \text{in}(S)$.*
(2) *D has a unique minimal quasi-kernel if and only if $V(D) = S \cup \text{in}(S) \cup \text{in}(\text{in}(S))$.*
(3) *If $R = V(D) - S - \text{in}(S) - \text{in}(\text{in}(S)) \neq \emptyset$ and $D[R]$ has no sinks, no 2-cycles and no 4-cycles, then D has at least three minimal quasi-kernels.*

Proof. Part (1) is Theorem 2.1 from [3] and part (2) is Theorem 2 from [1] restated for the converse digraph. For part (3), if K_1, K_2 and K_3 are the distinct minimal quasi-kernels for $D[R]$ provided by our Theorem 1, then clearly $S \cup K_1$, $S \cup K_2$ and $S \cup K_3$ are distinct minimal quasi-kernels for D. □

Comment. *Part (3) of Theorem 2 can be generalized as follows. For $\emptyset \neq W \subseteq V(D)$, define $S(W) =$ the set of sinks of $D[W]$, and $R(W) = W - S(W) - \text{in}(S(W)) - \text{in}(\text{in}(S(W)))$, (where we set $\text{in}(\emptyset) = \emptyset$). Now, let $V_1 = V(D)$ and, for $i \geq 1$, $V_{i+1} = R(V_i)$. This sequence of sets terminates with the smallest n such that either $V_n = \emptyset$, or $V_n \neq \emptyset$, but $S(V_n) = \emptyset$. In the latter case, if $D[V_n]$ has no 2-cycles and no 4-cycles, Theorem 1 applied to $D[V_n]$ gives distinct minimal quasi-kernels K_1, K_2, K_3. It is clear that $K_j \cup (\cup_{i=1}^{n-1} S(V_i))$ is a minimal quasi-kernel of D for $j = 1, 2, 3$.*

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